Monografías Matemáticas García de Galdeano **36**, 79–88 (2010)

A BIVARIATE STOCHASTIC GAMMA DIFFUSION MODEL: STATISTICAL INFERENCE

R. Gutiérrez, R. Gutiérrez-Sánchez, A. Nafidi

Abstract. In the present study, we propose a bivariate stochastic Gamma diffusion model as the solution to Ito's stochastic differential equations (SDE) that are similar as regards the drift and diffusion coefficients to those considered in the univariate Gamma diffusion model (see [11]). Firstly, we determine the main probabilistic characteristics of this model, such as the solution to the SDE, the bivariate transition density, the bidimensional moment functions, the conditioned trend functions and in particular, the correlation function between each of the components of the model. Then, based on some results of matrix differential calculus (see [13]), the statistical inference in the model is drawn, estimating the bidimensional drift and the diffusion matrix by the maximum likelihood method using discrete sampling. Finally, we obtain the properties of the resulting likelihood estimators.

Keywords: Bivariate stochastic Gamma diffusion process, Likelihood estimation using discrete sampling, Matrix differential calculus, Normal and Wishart random matrices.

AMS classification: 60J60, 62M05.

§1. Introduction

Stochastic diffusion processes are of great interest to investigators in many fields, such as biology, physics, demography, economics and environmental sciences. One of the questions that has aroused greatest interest about these stochastic models (especially in the one dimensional case) and one that has been the object of numerous studies in recent years, is that of statistical estimation and inference. Various methods addressing the question of statistical inference have been developed recently, and several papers published on the topic, including those by Biby and Sorensen [4] and Ait-Sahalia [1], without overlooking the wide-ranging review of results presented by Prakasa Rao [14], who provides a lengthy list of references on the subject.

As regards the statistical inference in multivariate diffusion processes, some special cases have been studied, such as the multivariate lognormal and Gompertz diffusion processes. An extensive study of the probabilistic aspects and of the corresponding statistical inference (estimation and test of hypotheses) can be seen, for example, for the first process in [6], [7] and [8] and for the second process in [9] and [5].

The main aim of this study is to propose and examine a "bivariate Gamma diffusion model". To do so, we follow the methodology previously employed to extend one-dimensional diffusion processes to the multivariate case. For example, in [6] and [7] this methodology is applied to the case of a multivariate lognormal diffusion model, and in [9] for the case of the bivariate Gompertz diffusion model. Specifically, in the present study, we extend the univariate stochastic Gamma diffusion process studied in [10] and [11] to a bivariate Gamma process. The latter process is constituted of two components, which are Gamma univariate processes and which are interrelated in the sense that they vary in a correlated way in their stochastic evolution in time. Having established this model, we then study its probabilistic characteristics and its associated basic statistical inference.

This paper is organized as follows. In the second section, we determine the main probabilistic characteristics of the model, such as the solution to the SDE, the bivariate transition density, the bidimensional moment functions, the conditioned trend functions and in particular, the correlation function between each of the components of the model. In the third section, the statistical inference in the model is achieved, and the bidimensional drift and the diffusion matrix are estimated by the maximum likelihood method based on discrete sampling. In the last section, based on some results of the matrix normal distribution, we obtain the properties of the resulting likelihood estimators.

§2. Bivariate SGDP and its characteristics

2.1. The model and its analytical expression

Let $\{x(t) = (x_1(t), x_2(t))'; t \in [t_0, T]; t_0 > 0\}$ be a bivariate stochastic process that satisfies the following Ito's SDE:

$$dx(t) = a(t, x(t))dt + b(t, x(t))dw(t) \quad ; \quad x(t_0) = x_{t_0} \tag{1}$$

with the vector $a(t, x)$ and the matrix $b(t, x)$ being given as follows

$$a(t, x) = D(x)\left(\frac{a}{t} - \beta\right) \quad ; \quad b(t, x) = D(x)B^{1/2}$$

where $\{w(t); t \in [t_0, T]\}$ is a 2-dimensional standard Wiener process, x_{t_0} is a fixed vector belonging to $(0, \infty)^2$, $x = (x_1, x_2)' \in (0, \infty)^2$, $a = (a_1, a_2)'$ and $\beta = (\beta_1, \beta_2)'$, in which $D(x)$ is a diagonal matrix where the elements of the principal diagonal are x_1, x_2, and where $B = (b_{ij})_{i,j}$ is a 2×2 symmetric non negative definite matrix. The parameters $a_1, a_2, \beta_1, \beta_2$ and $b_{i,j}$ for $1 \leq i, j \leq 2$ are real and will be the object of subsequent statistical estimation.

The vector $a(t, x)$ and the matrix $b(t, x)$ specified in Eq.(2) satisfy the Lipschitz restriction on growth conditions for the existence and unicity of the solution to the SDEs in theorem (6.2.2 page 105) of Arnold [3]. Thus, let:

$$K_1 = \max_{1 \leq i \leq 2}\left[\max_{t \in [t_0, T]}\left(\frac{a_i}{t} - \beta_i\right)^2\right] \quad \text{and} \quad K_2 = \max_{1 \leq i \leq 2}(b_{ii}).$$

Then, on the one hand, $\exists K = K_1^{1/2} + K_2^{1/2}, \forall t \in [t_0, T]$, such that $\forall x, y \in (0, \infty)^2$ we have

$$\|a(t,x) - a(y,t)\|_e + \|b(t,x) - b(t,y)\|_{tr} \leq K\|x-y\|_e$$
$$\|a(t,x)\|_e^2 + \|b(t,x)\|_{tr}^2 \leq K(1 + \|x\|_e^2)$$

where $\| \quad \|_e$ denotes the euclidean norm in $\mathbb{R}^2$ and $\| \quad \|_{tr}$ denotes the trace norm in $\mathcal{M}_{2\times 2}$ ($\|A\|_{tr} = [tr(AA')]^{1/2}$).

Then, under these conditions equation Eq.(1) has on $[t_0, T]$ a unique $\mathbb{R}^2$-valued solution $\{x(t); t \in [t_0, T]\}$, continuous with probability 1, that satisfies the initial condition $x(t_0) = x_{t_0}$.

On the other hand, as the vector $a(t,x)$ is a continuous function with respect to t, then by theorem (9.3.1 page 152) of Arnold [3], the solution $\{x(t); t \in [t_0, T]\}$ is a 2-dimensional diffusion process on $[t_0, T]$ with drift vector $a(t,x)$ and with a diffusion matrix given by

$$B(x) = (D(x)B^{1/2})(D(x)B^{1/2})' = D(x)BD(x) = (b_{ij}x_ix_j)_{1\leq i,j\leq 2}\,.$$

The analytical expression of process $\{x(t), t \in [t_0, T]\}$ can be obtained by applying Ito's formula (see, for example [3]) to a transform of the type $y(t) = \log(x(t)) = (\log(x_1(t)), \log(x_2(t)))'$, and then we obtain

$$dy(t) = \left[\frac{a}{t} - \left(\beta + \frac{b}{2}\right)\right] dt + B^{1/2}dw(t) \quad , \quad y(t_0) = \log(x_{t_0})$$

where $b = (b_{11}, b_{22})'$, and then by integration we have

$$y(t) = y(t_0) + \log\left(\frac{t}{t_0}\right) a - \left(\beta + \frac{b}{2}\right)(t - t_0) + B^{1/2}(w_t - w_{t_0})$$

from which we can deduce that the solution to the original SDE Eq. (1) has the following form

$$x(t) \quad = \quad \exp\left(\log(x_{t_0}) + \log\left(\frac{t}{t_0}\right) a - \left(\beta + \frac{b}{2}\right)(t - t_0) + B^{1/2}(w_t - w_{t_0})\right).$$

2.2. The ptdf and moments of the model

Taking into account that the random vector $(w(t) - w(s))$ has a bivariate normal distribution $\mathcal{N}_2\left(0, (t-s)I_2\right)$ (where I_2 denotes the 2×2 identity matrix), it can be deduced that $x(t) \mid x(s) = x_s$ has a bivariate lognormal distribution $\Lambda_2\left(\mu(s,t,x_s), (t-s)B\right)$ where $\mu(s,t,x_s)$ is the following 2-dimensional vector

$$\mu(s,t,x) \quad = \quad \log(x) + a\log\left(\frac{t}{s}\right) - \left(\beta + \frac{b}{2}\right)(t-s) \tag{2}$$

and therefore the transition density function of the process $f(y,t \mid x,s)$ (for $y = (y_1,y_2)'$ and $x = (x_1,x_2)'$) has the form

$$f(y,t \mid x,s) = [2\pi]^{-1}(t-s)^{-1} \mid B \mid^{-\frac{1}{2}} (y_1y_2)^{-1} \exp\left\{-\frac{Q}{2}\right\} \tag{3}$$

where $\mid B \mid$ is the determinant of the matrix B, and Q is a quadratic form that is given by

$$Q \quad = \quad (\log(y) - \mu(s,t,x))' \left[(t-s)B\right]^{-1} (\log(y) - \mu(s,t,x))$$

where $\mu(s,t,x)$ is as given in Eq. (2).

The marginal conditional and non-conditional moments of order r ($r \in \mathbb{N}^*$) can be obtained from the function generating the random vector $Z(t) = \log\left[x(t) \mid x(s) = x_s\right]$, which follows the law $\mathcal{N}_2\left(\mu(s,t,x_s);(t-s)B\right)$, and is expressed as follows, for $\lambda \in \mathbb{R}^2$

$$\mathbb{E}(e^{\lambda' Z(t)}) \quad = \quad \exp\left\{\lambda'\mu(s,t,x_s) + \frac{t-s}{2}\lambda' B\lambda\right\}.$$

For particular values of the vector $\lambda = (0,r)'$ or $\lambda = (r,0)'$ ($r \in \mathbb{N}^*$), we obtain, for example, the marginal conditional trend functions of order r of the process and which have the following form, for $i = 1,2$

$$\mathbb{E}\left(x_i^r(t) \mid x_i(s) = x_{s,i}\right) \quad = \quad \exp\left(r\mu_i(s,t,x_s) + \frac{r^2(t-s)}{2}b_{ii}\right) \tag{4}$$

and for $\lambda = (r_1,r_2)'$ ($r_1,r_2 \in \mathbb{N}^*$), we obtain the joint conditional trend of the process

$$\begin{aligned}\mathbb{E}\left(x_1^{r_1}(t)x_2^{r_2}(t) \mid x(s) = x_s\right) \quad = \quad & \exp\left(r_1\mu_1(s,t,x_s) + r_2\mu_2(s,t,x_s)\right. \\ & \left. + \frac{(t-s)}{2}(r_1^2b_{11} + r_2^2b_{22} + 2r_1r_2b_{12})\right).\end{aligned} \tag{5}$$

Using Eq. (4) in the particular case $r = 1$, we obtain the marginal conditional trend function of the process

$$\mathbb{E}\left(x_i(t) \mid x_i(s) = x_{s,i}\right) \quad = \quad \exp\left(\mu_i(s,t,x_s) + \frac{1}{2}(t-s)b_{ii}\right). \tag{6}$$

By assuming the initial condition $\mathrm{P}(x(t_0) = x_{t_0}) = 1$, and using Eq. (6) then the non conditional marginal trend functions are

$$\begin{aligned}\mathbb{E}(x_i(t)) &= \exp\left(\mu_i(t_0, t, x_{t_0}) + \frac{1}{2}(t - t_0)b_{ii}\right) \\ &= \frac{x_{t_0,i}e^{\beta_i t_0}}{t_0^{a_i}} t^{a_i} e^{-\beta_i t}.\end{aligned}$$

From Eq. (4) and Eq. (6), we can deduce that the marginal variance function of the process, for $i = 1, 2$ is:

$$\operatorname{Var}(x_i(t)) = \exp\left(2\mu_i(t_0, t; x_{t_0}) + (t - s)b_{ii}\right)\left(e^{(t-s)b_{ii}} - 1\right)$$

and the covariance function at the same instant is

$$\begin{aligned}\operatorname{Cov}(x_1(t), x_2(t)) &= \exp\left(\mu_1(t_0, t; x_{t_0}) + \mu_2(t_0, t; x_{t_0}) + \frac{1}{2}(t - s)(b_{11} + b_{22})\right) \\ &\quad \left(e^{(t-s)b_{12}} - 1\right).\end{aligned}$$

The correlation function of the process at the same instant is given by

$$\varrho(x_1(t), x_2(t)) = \frac{\left(e^{(t-t_0)b_{12}} - 1\right)}{\left(e^{(t-t_0)b_{11}} - 1\right)^{1/2}\left(e^{(t-t_0)b_{22}} - 1\right)^{1/2}}.$$

§3. Statistical inference on the model

3.1. Parameter likelihood estimation

Let us now obtain the maximum likelihood estimators of the parameters corresponding to the model β, a and B, using discrete sampling. To construct the likelihood function associated with the process, the following discrete sampling is used: $\{x(t_1) = x_{t_1}; x(t_2) = x_{t_2}; \ldots, x(t_n) = x_{t_n}\}$ at the instants $t_1, t_2; \ldots; t_n$, in which each $x(t_\alpha)$ represents the bidimensional vector $x(t_\alpha) = (x_1(t_\alpha), x_2(t_\alpha))'$, which for the sake of simplicity we shall denote as $x_{t_\alpha} = x_\alpha$. We also considered the initial condition $\mathrm{P}[x(t_1) = x_1] = 1$; by applying the Markov property and making use of Eq. (3), the likelihood function associated with the sample considered, of size $(n - 1)$ is given by

$$\begin{aligned}&\mathbb{L}(x_1, \ldots, x_n) = \prod_{\alpha=2}^{n} f(x_\alpha, t_\alpha \mid x_{\alpha-1}, t_{\alpha-1}) \\ &= (2\pi)^{-\frac{k(n-1)}{2}} \mid B \mid^{-\frac{(n-1)}{2}} \prod_{\alpha=2}^{n} (t_\alpha - t_{\alpha-1}) \left(\prod_{i=1}^{k} x_{\alpha,i}^{-1}\right) \\ &\exp\left\{-\frac{1}{2}\left[\log(x_\alpha/x_{\alpha-1}) - a\log(t_\alpha/t_{\alpha-1}) + \left(\beta + \frac{b}{2}\right)(t_\alpha - t_{\alpha-1})\right]' \right. \\ &\left. (t_\alpha - t_{\alpha-1})^{-1}B^{-1}\left[\log(x_\alpha/x_{\alpha-1}) - a\log(t_\alpha/t_{\alpha-1}) + \left(\beta + \frac{b}{2}\right)(t_\alpha - t_{\alpha-1})\right]\right\}.\end{aligned}$$

By carrying out the following change of variable: $v_1 = x_1$ and $v_\alpha = (t_\alpha - t_{\alpha-1})^{-1/2}$ $(\log(x_\alpha) - \log(x_{\alpha-1}))$ for $\alpha = 2, \ldots, n$, then, in terms of v_α, the likelihood function is given by

$$\mathbb{L}_{\mathbf{v}_1,\ldots,\mathbf{v}_n}(\Gamma; B) = (2\pi)^{-(n-1)k/2} \mid B \mid^{-\frac{(n-1)}{2}} \exp\left\{-\frac{1}{2}\sum_{\alpha=2}^{n} (\mathbf{v}_\alpha - \Gamma u_\alpha)' B^{-1} (\mathbf{v}_\alpha - \Gamma u_\alpha)\right\} \quad (7)$$

where, $u_\alpha = (t_\alpha - t_{\alpha-1})^{-1/2} (\log(t_\alpha/t_{\alpha-1}), t_\alpha - t_{\alpha-1})'$, for $\alpha = 2, \ldots, n$, and $\Gamma = \left(a, -\left(\beta + \frac{b}{2}\right)\right)$ is (2×2)- matrix.

Let $\mathbf{V} = (v_2, \ldots, v_n)$ and $\mathbf{U} = (u_2, \ldots, u_n)$. The likelihood function can then be written as follows:

$$\mathbb{L}_{\mathbf{V}} = (2\pi)^{-(n-1)} \mid B \mid^{-\frac{(n-1)}{2}} \exp\left\{-\frac{1}{2} tr \left[B^{-1} (\mathbf{V} - \Gamma\mathbf{U}) (\mathbf{V} - \Gamma\mathbf{U})'\right]\right\}.$$

By taking the logarithm, we obtain

$$\log(\mathbb{L}_{\mathbf{V}}) = -(n-1)\frac{k}{2}\log(2\pi) - \frac{n-1}{2}\log \mid B \mid -\frac{1}{2} tr \left[B^{-1} (\mathbf{V} - \Gamma\mathbf{U}) (\mathbf{V} - \Gamma\mathbf{U})'\right].$$

Then, calculating the differential of this function, and making use of the following results of matrix differential calculus (see, for example [13]): $d\,[\mathrm{tr}(B)] = \mathrm{tr}\,(dB)$, $d\,[\log \mid B \mid] = \mathrm{tr}\left(B^{-1}dB\right)$ and $d\left[B^{-1}\right] = B^{-1}(dB)B^{-1}$, we have

$$\begin{aligned} d\log(\mathbb{L}) = & -\frac{n-1}{2}\mathrm{tr}\left(B^{-1}dB\right) - \frac{1}{2} tr \left[-B^{-1}(dB)B^{-1} (\mathbf{V} - \Gamma\mathbf{U}) (\mathbf{V} - \Gamma\mathbf{U})'\right] \\ & -\frac{1}{2} tr \left[B^{-1} (-d\Gamma) \mathbf{U} (\mathbf{V} - \Gamma\mathbf{U})' + B^{-1} (\mathbf{V} - \Gamma\mathbf{U}) \mathbf{U}' (-d\Gamma)'\right]. \end{aligned}$$

By applying trace properties, the above differential can be written as follows:

$$\begin{aligned} d\log(\mathbb{L}) = & \frac{1}{2}\mathrm{tr}\left\{\left[B^{-1} (\mathbf{V} - \Gamma\mathbf{U}) (\mathbf{V} - \Gamma\mathbf{U})' - (n-1)I_2\right] B^{-1}dB\right\} \\ & + tr\left\{\mathbf{U} (\mathbf{V} - \Gamma\mathbf{U})' B^{-1}d\Gamma\right\}. \end{aligned}$$

From the relations $tr(AB) = Vec'(A')Vec(B)$ and $dVec(A) = Vec(dA)$, where Vec denotes the matrix vectorization (given an $n \times m$ matrix X, the $Vec(X)$ is the vector of dimension $nm \times 1$ that stacks the columns of X), we obtain

$$\begin{aligned} d\log(\mathbb{L}) = & \frac{1}{2}Vec'\left\{\left[B^{-1} (\mathbf{V} - \Gamma\mathbf{U}) (\mathbf{V} - \Gamma\mathbf{U})' - (n-1)I_2\right] B^{-1}\right\} d\mathrm{Vec}(B) \\ & + Vec'\left\{B^{-1} (\mathbf{V} - \Gamma\mathbf{U}) \mathbf{U}'\right\} d\mathrm{Vec}(\Gamma). \end{aligned}$$

Then, making this differential equal to zero, with respect to the estimators of B and γ, we obtain

$$B^{-1}(\mathbf{V}-\Gamma\mathbf{U})\,\mathbf{U}'=0 \tag{8}$$

$$\left[B^{-1}(\mathbf{V}-\Gamma\mathbf{U})(\mathbf{V}-\Gamma\mathbf{U})'-(n-1)I_2\right]B^{-1}=0. \tag{9}$$

From Eq. (8) and Eq. (9), we obtain the maximum likelihood estimators of the matrices Γ and B, which are given by

$$\widehat{\Gamma}=\mathbf{VU}'\left(\mathbf{UU}'\right)^{-1} \tag{10}$$

$$(n-1)\widehat{B}=\mathbf{VH_U V}' \tag{11}$$

where $\mathbf{H_U}=I_{n-1}-\mathbf{U}'\left(\mathbf{UU}'\right)^{-1}\mathbf{U}$ is an idempotent symmetric matrix.

3.2. Likelihood estimator distribution

In order to study the estimator distributions obtained by the expressions Eq. (10) and Eq. (11), it is necessary to examine some results of the matrix normal distribution (see, for example [13]), which are presented as follows:

Definition 1. : Let $X_{m\times n}$ be a random matrix and let $M_{m\times n}$, $C_{m\times m}$ and $D_{n\times n}$ be constant matrices (C and D are non negative definite matrices). We then say that the random matrix X has a normal distribution and it is denoted by $\mathcal{N}_{m\times n}(M; C\otimes D)$ ($\otimes$ denotes the Kronecker product of matrices: $C\otimes D=[c_{ij}D]$) if the density function of X is

$$f(x)=(2\pi)^{-\frac{mn}{2}}\mid C\mid^{-\frac{m}{2}}\mid D\mid^{-\frac{n}{2}}\exp\left\{-\frac{1}{2}tr\left[C^{-1}(M-x)D^{-1}(M-x)'\right]\right\}.$$

In the sense of the matrix vectorization, we have the following equivalence:

$$X\sim\mathcal{N}_{m\times n}(M;C\otimes D)\ \text{ if only if } vec(X)\sim\mathcal{N}_{mn}(vec(M');C\otimes D).$$

Corollary 1. *: Let $X\sim\mathcal{N}_{m\times n}(M;C\otimes D)$ and let $N_{p\times m}$ be a constant matrix. We then have*

$$X'\sim\mathcal{N}_{n\times m}(M';D\otimes C).$$

$$NX\sim\mathcal{N}_{n\times p}(NM;NCN'\otimes D).$$

3.2.1. Distribution of $\widehat{\Gamma}$

The expression in Eq.(7) can be rewritten as follows

$$\mathbb{L}_{\mathbf{V}}=(2\pi)^{-(n-1)}\mid B\mid^{-\frac{n-1}{2}}\mid I_{n-1}\mid^{-1}\exp\left\{-\frac{1}{2}tr\left[B^{-1}(\mathbf{V}-\Gamma\mathbf{U})I_{n-1}^{-1}(\mathbf{V}-\Gamma\mathbf{U})'\right]\right\}.$$

From which we deduce the matrix

$$\mathbf{V}\sim\mathcal{N}_{2\times(n-1)}(\Gamma\mathbf{U};B\otimes\mathbf{I}_{n-1}).$$

Then, by using Corollary 1, we have

$$\mathbf{VU}'\left(\mathbf{UU}'\right)^{-1} \sim \mathcal{N}_{2\times 2}\left(\Gamma\mathbf{UU}'\left(\mathbf{UU}'\right)^{-1}; B\otimes\left(\mathbf{UU}'\right)^{-1}\mathbf{U}\mathrm{I}_{n-1}\mathbf{U}'\left(\mathbf{UU}'\right)^{-1}\right).$$

Thus, we obtain that

$$\widehat{\Gamma} \sim \mathcal{N}_{2\times 2}\left(\Gamma, B\otimes\left(\mathbf{UU}'\right)^{-1}\right).$$

3.2.2. Distribution of $\widehat{B}$

To obtain the distribution of the matrix $\widehat{B}$, we make use of the following result (see for example [15], corollary 3.2):

Corollary 2. *If $Y \sim \mathcal{N}_{n\times p}[\mu, A\otimes\Sigma]$, then $Y'WY$ has a non central Wishart distribution with m degrees of freedom, covariance Σ and noncentral matrix λ noted by $\mathcal{W}_p(m,\Sigma,\lambda)$, if and only if:*

$$AWAWA = AWA, \qquad tr(AW) = m$$
$$\lambda = \mu' W\mu = \mu' WAW\mu = \mu' WAWAW\mu.$$

Using the latter result in the particular case: $Y = V'$, $A = \mathrm{I}_{n-1}$, $\Sigma = B$, $W = \mathrm{H}_{\mathrm{U}}$ and $\mu = U'\Gamma'$, and so we have: $\mathrm{tr}(AW) = m = n-3$ and $\lambda = 0$, and $V\mathrm{H}_{\mathrm{U}}V' \sim \mathcal{W}_2(n-3, B)$ and therefore by symmetric properties of the Wishart distribution, we deduce that

$$(n-1)\hat{B} \sim \mathcal{W}_2(n-3, B).$$

3.2.3. Independence of likelihood estimators

To show that $\widehat{\Gamma}$ and $\hat{B}$ are independently distributed, we make use of the following independence result between linear and quadratic forms (see for example [12] corollary 6):

Corollary 3. *Let $Y \sim \mathcal{N}_{n\times p}[\mu, A\otimes\Sigma]$, Then, the necessary and sufficient conditions for the independence of $YWY' + \frac{1}{2}(LY' + YL' + C)$ and YM' are $AWM' = 0$ and $LWM' = 0$.*

By applying this result to the particular case $Y = V$, $W = \mathrm{H}_{\mathrm{U}}$, $A = B$, $\Sigma = I_{n-1}$, $L = 0$, $C = 0$ and $M' = \mathbf{U}'\left(\mathbf{UU}'\right)^{-1}$, the necessary and sufficient conditions for independence are satisfied and we have established that $\hat{\Gamma}$ and $\hat{B}$ are independently distributed.

3.3. Sufficiency and completeness

We have $(\mathbf{V}-\Gamma\mathbf{U})(\mathbf{V}-\Gamma\mathbf{U})' = \left([\mathbf{V}-\widehat{\Gamma}\mathbf{U}] + [\widehat{\Gamma}-\Gamma]\mathbf{U}\right)\left([\mathbf{V}-\widehat{\Gamma}\mathbf{U}] + [\widehat{\Gamma}-\Gamma]\mathbf{U}\right)'$. Then, by developing and using Eq.(11), we obtain:

$$\begin{aligned}(\mathbf{V}-\Gamma\mathbf{U})(\mathbf{V}-\Gamma\mathbf{U})' &= \left(\mathbf{V}-\widehat{\Gamma}\mathbf{U}\right)\left(\mathbf{V}-\widehat{\Gamma}\mathbf{U}\right)' + \left(\widehat{\Gamma}-\Gamma\right)\mathbf{UU}'\left(\widehat{\Gamma}-\Gamma\right)' \\ &= (n-1)\widehat{B} + \left(\widehat{\Gamma}-\Gamma\right)\mathbf{UU}'\left(\widehat{\Gamma}-\Gamma\right)'.\end{aligned}$$

The latter equation can be written as:

$$\begin{aligned}\mathbb{L}_{\mathbf{V}}(\Gamma, B) &= (2\pi)^{-(n-1)} \mid B \mid^{-\frac{n-1}{2}} \\ &\exp\left\{-\frac{1}{2}tr\left[B^{-1}\left((n-1)\widehat{B} + \left(\widehat{\Gamma} - \Gamma\right)\mathbf{UU}'\left(\widehat{\Gamma} - \Gamma\right)'\right)\right]\right\}\end{aligned}$$

which means that $\left(\widehat{\Gamma}, \widehat{B}\right)$ is conjointly sufficient for (Γ, B)

The completeness of $\left(\widehat{\Gamma}, \widehat{B}\right)$ follows, by reasoning similar to that employed for the maximum likelihood estimators of the parameters of the multivariate normal distribution (see, for example, Anderson [2]).

Finally, as the estimators $\widehat{\Gamma}$ and $\frac{n-1}{n-3}\hat{B}$ are unbiased for Γ and B respectively, then we deduce that they are the UMVUE.

Acknowledgements

This work was supported partially by research project MTM 2005-09209, Ministerio de Educación y Ciencia, and project P06-FQM-02271, Junta de Andalucía, Spain.

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Nafidi A.
Université Hassan 1er
Ecole Supérieure de Technologie- Berrechid
B.P: 218, Berrechid, Maroc
nafidi@estb.ac.ma

Gutiérrez R. and Gutiérrez-Sánchez R.
Department of Statistics and Operations Research
University of Granada, Facultad de Ciencias
Campus de Fuentenueva s/n, 18071
Granada, Spain
rgjaimez@ugr.es and
ramongs@ugr.es